

ON A FAT SMALL OBJECT ARGUMENT

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ABSTRACT. Good colimits introduced by J. Lurie generalize transfinite composites and provide an important tool for understanding cofibrant generation in locally presentable categories. We will explore the relation of good colimits to transfinite composites further and show, in particular, how they eliminate the use of large objects in the usual small object argument.

1. INTRODUCTION

Combinatorial model categories were introduced by J. H. Smith as model categories which are locally presentable and cofibrantly generated. The latter means that both cofibrations and trivial cofibrations are cofibrantly generated by a set of morphisms. He has not published his results but most of them can be found in [2], [3], [11] and [15]. A typical feature of a combinatorial model category \mathcal{K} is the existence of a regular cardinal λ such that everything happens below λ , i.e., among λ -presentable objects, and then it is extended to \mathcal{K} by using λ -filtered colimits. In particular, fibrant objects form a λ -accessible category and any cofibrant object is a λ -filtered colimit of λ -presentable cofibrant objects. In general, this cardinal λ is greater than κ in which \mathcal{K} is presented. This means that \mathcal{K} is κ -combinatorial in the sense that it is locally κ -presentable and both cofibrations and trivial cofibrations are cofibrantly generated by a set of morphisms between κ -presentable objects. For example, the model category **SSet** of simplicial sets is ω -combinatorial but finitely presentable simplicial sets have ω_1 -presentable fibrant replacements. One of our main results is that any cofibrant object in a κ -combinatorial model category is a

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κ -filtered colimit of κ -presentable cofibrant objects. The proof is based on the concept of a good colimit. Good colimits were introduced by Lurie in [11] and studied by the first author in [12]. They generalize transfinite composites but, while transfinite composites are thin and include large objects, good colimits are fat but their objects can be made small. This leads to the just mentioned result and may be called a *fat small object argument*. One of its consequences is the result of Joyal and Wraith [9] that any acyclic simplicial sets is a filtered colimit of finitely presentable acyclic simplicial sets. The original motivation for the introduction of good limits in [11] was to prove that a retract of a cellular morphism is cellular in retracts (of small cellular morphisms).

It is remarkable that the same idea independently emerged in module theory where the Hill lemma was used for the same purpose (see [16]). In particular, [16] shows, in this additive setup, that a retract of a cellular morphism is cellular in retracts. Both the model category and the module theory situation subsumes into the framework of a locally presentable category equipped with a cofibrantly generated weak factorization system. We are working in this context and show how filtered colimits mix with those used in cofibrant generation. More results in this direction will be presented in [14].

In the appendix, we reprove Lurie's result about the elimination of retracts using κ -good colimits which are moreover κ -directed (this is a major departure from Lurie's approach). Such colimits play a central role in our paper, and are essential for our applications.

2. WEAK FACTORIZATION SYSTEMS

Let \mathcal{K} be a category and $f: A \rightarrow B$, $g: C \rightarrow D$ morphisms such that in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{v} & D \end{array}$$

there is a diagonal $d: B \rightarrow C$ with $df = u$ and $gd = v$. Then we say that g has the *right lifting property* w.r.t. f and f has the *left lifting property* w.r.t. g . For a class \mathcal{X} of morphisms of \mathcal{K} we put

$$\mathcal{X}^\square = \{g \mid g \text{ has the right lifting property w.r.t. each } f \in \mathcal{X}\} \text{ and}$$

$${}^\square\mathcal{X} = \{f \mid f \text{ has the left lifting property w.r.t. each } g \in \mathcal{X}\}.$$

A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category \mathcal{K} consists of two classes \mathcal{L} and \mathcal{R} of morphisms of \mathcal{K} such that

- (1) $\mathcal{R} = \mathcal{L}^\square$, $\mathcal{L} = {}^\square\mathcal{R}$, and
- (2) any morphism h of \mathcal{K} has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

A weak factorization system $(\mathcal{L}, \mathcal{R})$ is called *cofibrantly generated* if there is a set \mathcal{X} of morphisms such that $\mathcal{R} = \mathcal{X}^\square$.

Notation 2.1. In order to state closure properties of the class \mathcal{L} , we introduce the following notation. Let \mathcal{X} be a class of morphisms in \mathcal{K} .

- (1) $\text{Po}(\mathcal{X})$ denotes the class of pushouts of morphisms in \mathcal{X} : $f \in \text{Po}(\mathcal{X})$ iff f is an isomorphism or there is a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & \lrcorner & \uparrow \\ X & \xrightarrow{g} & Y \end{array}$$

with $g \in \mathcal{X}$.

- (2) $\text{Tc}(\mathcal{X})$ denotes the class of transfinite composites (= compositions) of morphisms from \mathcal{X} : $f \in \text{Tc}(\mathcal{X})$ iff there is a smooth chain $(f_{ij}: A_i \rightarrow A_j)_{i \leq j \leq \lambda}$ (i.e., λ is an ordinal, $(f_{ij}: A_i \rightarrow A_j)_{i < j}$ is a colimit for any limit ordinal $j \leq \lambda$) such that $f_{i,i+1} \in \mathcal{X}$ for each $i < \lambda$ and $f = f_{0\lambda}$.

- (3) $\text{Rt}(\mathcal{X})$ denotes the class of retracts of morphisms in \mathcal{X} in the category \mathcal{K}^2 of morphisms of \mathcal{K} .

- (4) $\text{cell}(\mathcal{X}) = \text{Tc Po}(\mathcal{X})$ denotes the *cellular closure* of \mathcal{X} ; the elements of $\text{cell}(\mathcal{X})$ are called *\mathcal{X} -cellular maps* or *relative \mathcal{X} -cell complexes*, and

- (5) $\text{cof}(\mathcal{X}) = \text{Rt Tc Po}(\mathcal{X})$ the *cofibrant closure* of \mathcal{X} ; the elements of $\text{cof}(\mathcal{X})$ are called *\mathcal{X} -cofibrations* or simply *cofibrations*.

A basic property of a locally presentable category \mathcal{K} is that the pair $(\text{cof}(\mathcal{X}), \mathcal{X}^\square)$ is a weak factorization system for any set \mathcal{X} of morphisms. In particular

$${}^\square(\mathcal{X}^\square) = \text{cof}(\mathcal{X})$$

(“small object argument”); see [2].

Later, we will use the following simple observation: in the above (defining) equality $\text{cof}(\mathcal{X}) = \text{Rt Tc Po}(\mathcal{X})$, it is sufficient to consider retractions whose domain components are the identity morphisms, i.e. retractions taking place in the respective under category A/\mathcal{K} . This is because any retract f of g can be expressed also as a retract of the

pushout g' of g along the retraction r of the domains,

$$\begin{array}{ccccc}
 B & \longrightarrow & Y & \xrightarrow{\quad} & Y' & \dashrightarrow & B \\
 \uparrow f & & \uparrow g & & \uparrow g' & & \uparrow f \\
 A & \longrightarrow & X & \xrightarrow{r} & A & &
 \end{array}$$

(6) Let \mathcal{K}_κ denote the full subcategory of \mathcal{K} consisting of κ -presentable objects and $(\mathcal{K}_\kappa)^2$ the category of morphisms of \mathcal{K}_κ . For a class $\mathcal{X} \subseteq (\mathcal{K}_\kappa)^2$, we put

$$\begin{aligned}
 \text{Po}_\kappa(\mathcal{X}) &= \text{Po}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2 \\
 \text{cell}_\kappa(\mathcal{X}) &= \text{cell}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2 \\
 \text{cof}_\kappa(\mathcal{X}) &= \text{cof}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2
 \end{aligned}$$

and denote $\kappa\text{-Tc}(\mathcal{X})$ the class of transfinite composites of length smaller than κ , i.e., $\lambda < \kappa$ in (2).

3. FINITE FAT SMALL OBJECT ARGUMENT

In this short section, we outline our fat small object argument in the case $\kappa = \aleph_0$. We assume that \mathcal{K} is locally finitely presentable and that the set \mathcal{X} consists of morphisms between finitely presentable objects. For simplicity, we will also assume that all \mathcal{X} -cofibrations are regular monomorphisms (this assumption can be removed).

Let $f: A \rightarrow B$ be a morphism. A presented finite cell complex (see [5, Section 10.6]) in $A/\mathcal{K}/B$ (i.e. the category of objects of \mathcal{K} under A and over B) is a finite sequence C of the form

$$A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_n \rightarrow B,$$

whose composition is f , together with an expression of each $A_{i-1} \rightarrow A_i$ as a pushout of a finite coproduct of elements of \mathcal{X} ,

$$\begin{array}{ccc}
 A_{i-1} & \longrightarrow & A_i \\
 \uparrow & & \uparrow \\
 \bigsqcup_{j \in J_i} X_j & \xrightarrow{\bigsqcup g_j} & \bigsqcup_{j \in J_i} Y_j
 \end{array}$$

We call the components $X_j \rightarrow A_{i-1}$ of the left vertical map the characteristic maps and assume that they are all different (no two cells are glued along the same map at the same step). We stress that the sets J_i and the characteristic maps are taken as a part of the structure of a presented cell complex. We denote $|C| = A_n$ the “total space” of C , it is naturally an object of $A/\mathcal{K}/B$. We also define $A_m = A_n$ for $m \geq n$. In this way a presented finite cell complex can be prolonged arbitrarily.

A subcomplex inclusion $\iota: C \rightarrow C'$ is a sequence of morphisms $\iota_i: A_i \rightarrow A'_i$ for which there exists a diagram

$$\begin{array}{ccccc} A_{i-1} & \longleftarrow & \bigsqcup_{j \in J_i} X_j & \longrightarrow & \bigsqcup_{j \in J_i} Y_j \\ \downarrow \iota_{i-1} & & \downarrow & & \downarrow \\ A'_{i-1} & \longleftarrow & \bigsqcup_{j \in J'_i} X_j & \longrightarrow & \bigsqcup_{j \in J'_i} Y_j \end{array}$$

with the two unlabelled vertical maps the inclusions of sub-coproducts, corresponding to $J_i \subseteq J'_i$, and the induced map on pushouts being ι_i . We denote by $|\iota| = \iota_n$ the top part of the sequence ι , where n is at least the length of C and C' .

The presented finite cell complexes together with their inclusions form a directed poset (this uses the assumption on cofibrations being regular monomorphisms — otherwise, it would not have been even a poset). For details, see [5, Section 10.6 and Chapter 12]. The total space functor thus provides a directed diagram in $A/\mathcal{K}/B$. We form its colimit — a factorization

$$A \rightarrow A' \rightarrow B.$$

of the map f . The first map can be seen to lie in $\text{cell}(\mathcal{X})$ (e.g. as a consequence of Proposition 4.5). Now, we will show that the second map lies in \mathcal{X}^\square . Given a commutative square

$$\begin{array}{ccc} X & \longrightarrow & A' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & B \end{array}$$

with the left map in \mathcal{X} , we use finite presentability of X to factor the morphism $X \rightarrow A' = \text{colim}_C |C|$ through some finite cell complex $|C|$ in the diagram. By prolonging the presentation of C by one step, we construct a new presented cell complex C' with $|C'| = |C| \sqcup_X Y$. The composition $Y \rightarrow |C'| \rightarrow A'$ is then a diagonal in the above square.

We call this the fat small object argument, as it does not express A' as a “long” transfinite composite, but rather as a colimit of a spread out diagram where all objects are obtained using a finite number of cells. Thus, when A and all domains and codomains of \mathcal{X} are finitely presentable, the same applies to all objects in this diagram and A' is a directed colimit of a digram of finitely presentable objects.

In the proceeding, we will characterize the “good” diagrams arising in the fat small object argument and show that their “composites” lie in $\text{cell}(\mathcal{X})$ in general. When one comes to uncountable κ , much

more care has to be taken. The diagram obtained from cell complexes with $< \kappa$ cells is not good anymore (it is still κ -directed and has the desired colimit). We will describe this general case in more detail in an appendix.

4. GOOD COLIMITS

Recall that a poset P is well-founded if every of its nonempty subsets contains a minimal element. Given $x \in P$, $\downarrow x = \{y \in P \mid y \leq x\}$ denotes the initial segment generated by x .

Definition 4.1. We say that a poset P is *good* if it is well-founded and has a least element \perp . A good poset is called κ -*good* if all its initial segments $\downarrow x$ have cardinality $< \kappa$.

Any well-ordered set is good and every finite poset with a least element is good; in particular, the shape poset for pushout, a three-element good poset which is not a chain. The following terminology is transferred from well-ordered sets.

An element x of a good poset P is called *isolated* if

$$\Downarrow x = \{y \in P \mid y < x\}$$

has a top element x^- which is called the *predecessor* of x . A non-isolated element distinct from \perp is called *limit*. Given $x < y$ in a poset P , we denote xy the unique morphism $x \rightarrow y$ in the category P .

Definition 4.2. A diagram $D: P \rightarrow \mathcal{K}$ is *smooth* if, for every limit $x \in P$, the diagram $(D(y): Dy \rightarrow Dx)_{y < x}$ is a colimit cocone on the restriction of D to $\Downarrow x$.

A *good* diagram $D: P \rightarrow \mathcal{K}$ is a smooth diagram whose shape category P is a good poset.

Example 4.3. The canonical diagram from Section 3 is ω -good.

The *links* in a good diagram $D: P \rightarrow \mathcal{K}$ are the morphisms $D(x^-x)$ for the isolated elements $x \in P$.

The following result can be found both in [12] and in [11], A.1.5.6. The proof is “the same” as for transfinite composites.

Proposition 4.4. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category \mathcal{K} and $D: P \rightarrow \mathcal{K}$ a good diagram with links in \mathcal{L} . Then all components of a colimit cocone $\delta_x: Dx \rightarrow \text{colim } D$ belong to \mathcal{L} .

Proof. Since the principal filter $\uparrow x$ is good for each $x \in P$, it suffices to show that the component $\delta_\perp : D \perp \rightarrow \text{colim } D$ belongs to \mathcal{L} . Consider

$$\begin{array}{ccc} D \perp & \xrightarrow{u} & X \\ \delta_\perp \downarrow & & \downarrow g \\ \text{colim } D & \xrightarrow{v} & Y \end{array}$$

with $g \in \mathcal{R}$. It suffices to construct a compatible cocone $d_x : Dx \rightarrow X$ such that $d_\perp = u$ and $gd_x = v\delta_x$ for each $x \in P$. Then the induced morphism $d : \text{colim } D \rightarrow X$ is the desired diagonal in the square above. Since P is well-founded, we can proceed by recursion. Assume that we have d_y for each $y < x$. If x is limit we get d_x as induced by the cocone $(d_y)_{y < x}$. If x is isolated we get d_x as the diagonal in the square

$$\begin{array}{ccc} Dx^- & \xrightarrow{d_{x^-}} & X \\ D(x^-x) \downarrow & & \downarrow g \\ Dx & \xrightarrow{v} & Y \end{array}$$

□

The *composite* of a good diagram $D : P \rightarrow \mathcal{K}$ is the component δ_\perp of a colimit cocone. A *good composite* of morphisms from \mathcal{X} is the composite of a good diagram with links in \mathcal{X} . The just proved proposition says that \mathcal{L} is closed under good composites. This proposition can be strengthened as follows.

Proposition 4.5. *Let \mathcal{X} be a class of morphisms in a cocomplete category \mathcal{K} . Then the composite of a good diagram in \mathcal{K} with links in $\text{Po}(\mathcal{X})$ belongs to $\text{cell}(\mathcal{X})$.*

Proof. Let $D : P \rightarrow \mathcal{K}$ be a good diagram with links in $\text{Po}(\mathcal{X})$. Let \preceq be a well-ordering of P extending its partial ordering \leq (see [4], 1.2, Theorem 5). Let Q consist of all non-empty initial segments of (P, \preceq) . Then (Q, \subseteq) is a well-ordered set. Consider the diagram $E : Q \rightarrow \mathcal{K}$ such that ES is a colimit of the restriction of D to S and $E_{SS'} : ES \rightarrow ES'$ is the induced morphism. It is easy to see that E is a smooth transfinite sequence and by definition, $\text{colim } D \cong EP$.

It remains to show that links of E belong to $\text{Po}(\mathcal{X})$. These links are precisely $E[x] \rightarrow E[x]$, where $[z] = \{y \in P \mid y \preceq z\}$ and $[z) = \{y \in P \mid y < z\}$. Treating both as subposets of P , we have a pushout diagram

of categories and an induced pushout diagram of colimits in \mathcal{K} :

$$\begin{array}{ccc}
 [x] & \longrightarrow & [x] \\
 \uparrow & \lrcorner & \uparrow \\
 \Downarrow x & \longrightarrow & \downarrow x
 \end{array}
 \qquad
 \begin{array}{ccc}
 E[x] & \xrightarrow{E([x][x])} & E[x] \\
 \uparrow & \lrcorner & \uparrow \\
 E(\Downarrow x) & \longrightarrow & E(\downarrow x)
 \end{array}$$

When x is isolated, the bottom map is $D(x^-x)$ and when x is limit, it is an isomorphism. In both cases $E([x][x])$ belongs to $\text{Po}(\mathcal{X})$. \square

The class of good composites of diagrams with links in \mathcal{X} will be denoted $\text{Gd}(\mathcal{X})$. Analogously, $\kappa\text{-Gd}(\mathcal{X})$ denotes κ -good composites with links in \mathcal{X} . A transfinite composite is κ -good if and only if it is of length $\leq \kappa$.

Remark 4.6. The proposition above can be refined as follows.

Let λ be an infinite cardinal. Then the composite of a κ -good diagram of cardinality $< \lambda$ with links in $\text{Po}(\mathcal{X})$ belongs to $\lambda\text{-TcPo}(\mathcal{X})$.

Moreover, if κ is regular and $\lambda \geq \kappa$, then this composite can be expressed as a transfinite composite of length exactly λ with links in $\text{Po}(\mathcal{X})$. This follows from the fact that the well-ordering of P from the proof of 4.5 can be chosen isomorphic to the ordinal λ (see [4], Theorem 5).

Notation 4.7. Let $D: P \rightarrow \mathcal{K}$ be a good diagram and Q an initial segment of P . Then $\text{colim}_Q D$ will denote the colimit of the restriction of D on Q .

Remark 4.8. As with most of our statements, there is also a *relative* version: given an initial segment $Q \subseteq P$ and a diagram $D: P \rightarrow \mathcal{K}$ such that the links $D(x^-x)$ lie in $\text{Po}(\mathcal{X})$ for all $x \in P \setminus Q$, the induced map on colimits $\text{colim}_Q D \rightarrow \text{colim} D$ belongs to $\text{cell}(\mathcal{X})$. The proof is the same, only with all the elements of Q ignored. A particularly simple case of this relative version is the following lemma.

Lemma 4.9. *Let $D: P \rightarrow \mathcal{K}$ be a good diagram and let $Q \subseteq P$ be an initial segment. If all the elements in $P \setminus Q$ are limit, the induced map $\text{colim}_Q D \rightarrow \text{colim} D$ is an isomorphism.* \square

Transfinite composites are thin and long and are used for a weak factorization of a morphism h . This procedure is called a “small object argument”. We will show how to convert a transfinite composite into a fat and short good composite. Our procedure can be called a *fat small object argument*.

First we will prove an auxiliary lemma.

Lemma 4.10. *Let $D: P \rightarrow \mathcal{K}$ be a κ -good diagram. Then there exists its extension $D^*: P^* \rightarrow \mathcal{K}$ to a κ -good κ -directed diagram. In this extension, $P \subseteq P^*$ is an initial segment and all the elements in $P^* \setminus P$ are limit. In particular, the links of D^* are exactly those of D and the natural map $\text{colim } D \rightarrow \text{colim } D^*$ is an isomorphism.*

Proof. We will construct P^* and D^* by iterating transfinitely the following construction. Let P^+ consists of adding, for each initial segment $S \subseteq P$ of cardinality $< \kappa$ without a greatest element, an element p_S such that $s < p_S$ for each $s \in S$. The added elements p_S are incomparable among themselves. The extension $D^+: P^+ \rightarrow \mathcal{K}$ is given by $D^+(p_S) = \text{colim}_S D$. Thus, there are no new links in P^+ and D^+ is still κ -good. Define inductively P^γ as $P^0 = P$, $P^{\gamma+1} = (P^\gamma)^+$ and $P^\gamma = \bigcup_{\eta < \gamma} P^\eta$ for a limit γ . The diagrams D^γ are defined in a similar fashion. We set $P^* = P^\kappa$ and $D^* = D^\kappa$. Since every subset of P^* of cardinality $< \kappa$ lies in some P^γ , $\gamma < \kappa$, it has an upper bound in $P^{\gamma+1}$. Consequently, P^* is κ -directed; it is still κ -good. \square

Theorem 4.11. *Let \mathcal{K} be a cocomplete category and \mathcal{X} a class of morphisms with κ -presentable domains. Then any morphism from $\text{cell}(\mathcal{X})$ is a composite of a κ -good κ -directed diagram with links in $\text{Po}(\mathcal{X})$.*

Proof. Let $f \in \text{cell}(\mathcal{X})$. There is a smooth chain $(f_{\beta\alpha}: A_\beta \rightarrow A_\alpha)_{\beta \leq \alpha \leq \lambda}$ with links in $\text{Po}(\mathcal{X})$ such that $f = f_{0\lambda}$. We will proceed by recursion and prove that each $f_{0\alpha}$, $\alpha \leq \lambda$ is a composite of a κ -good κ -directed diagram $D_\alpha: P_\alpha \rightarrow \mathcal{K}$ with links in $\text{Po}(\mathcal{X})$. Moreover, D_β is the restriction of D_α on the initial segment $P_\beta \subseteq P_\alpha$ for each $\beta < \alpha \leq \lambda$. We put $P_\alpha = \alpha + 1$ for $\alpha < \kappa$, $P_\kappa = \kappa$ and, in both cases, D_α is the restriction of our chain to P_α , $\alpha \leq \kappa$. Let $\kappa < \alpha$ and assume that the claim holds for each $\beta < \alpha$.

Let $\alpha = \beta + 1$. Then $f_{\beta\alpha}$ is a pushout

$$\begin{array}{ccc} A_\beta & \xrightarrow{f_{\beta\alpha}} & A_\alpha \\ u \uparrow & \lrcorner & \uparrow v \\ X & \xrightarrow{g} & Y \end{array}$$

with g in \mathcal{X} . Since X is κ -presentable and D_β is κ -directed, the morphism $u: X \rightarrow A_\beta \cong \text{colim } D_\beta$ factors through some $u_x: X \rightarrow D_{\beta x}$.

For $x \leq y$, we denote $u_y = D_\beta(xy)u_x$. Take pushouts

$$\begin{array}{ccc} D_\beta y & \xrightarrow{f_{\beta y}} & A_y \\ u_y \uparrow & \lrcorner & \uparrow v_y \\ X & \xrightarrow{g} & Y \end{array}$$

By adding to the diagram D_β the objects A_y , for $x \leq y$, and the obvious morphisms $f_{\beta y}: D_\beta y \rightarrow A_y$ and $A_y \rightarrow A_z$, for $x \leq y < z$, we obtain a κ -good κ -directed diagram D_α . Clearly, $P_\beta \subseteq P_\alpha$ is an initial segment with a single new isolated element corresponding to A_x . The colimit of this diagram is A_α .

Let α be a limit ordinal and Q the union of P_β , $\beta < \alpha$. Since, for $\gamma < \beta < \alpha$, $P_\gamma \subseteq P_\beta$ is an initial segment, this union Q is κ -good but not necessarily κ -directed. Denoting by $E: Q \rightarrow \mathcal{K}$ the union of D_β , $\beta < \alpha$, we define $P_\alpha = Q^*$ and $D_\alpha = E^*$ using the previous lemma. The links of D_α are those of E , i.e. those of D_β , $\beta < \alpha$, and, in particular, they lie in $\text{Po}(\mathcal{X})$. The colimit of D_α is $\text{colim } E = \text{colim}_{\beta < \alpha} \text{colim } D_\beta = \text{colim}_{\beta < \alpha} A_\beta = A_\alpha$. \square

Let $\kappa\text{-GdDir}(\mathcal{X})$ denote the collection of all κ -good κ -directed composites with links in \mathcal{X} .

Corollary 4.12. *Let \mathcal{K} be a cocomplete category and \mathcal{X} a class of morphisms in \mathcal{K}_κ . Then $\text{cell}(\mathcal{X}) = \kappa\text{-GdDir } \text{Po}(\mathcal{X})$.*

Proof. It follows from Proposition 4.4 and Theorem 4.11. \square

Corollary 4.13. *Let \mathcal{K} be a cocomplete category and \mathcal{X} a class of morphisms in \mathcal{K}_κ . Then a morphism with the domain in \mathcal{K}_κ belongs to $\text{cell}(\mathcal{X})$ if and only if it belongs to $\kappa\text{-GdDir } \text{Po}_\kappa(\mathcal{X})$.*

Proof. Let $D: P \rightarrow \mathcal{K}$ be a κ -good diagram with a κ -presentable $D \perp$. Then all objects in the diagram D are κ -presentable too: this can be seen by an easy induction on the well-founded partial ordering on P . The rest follows from Corollary 4.12. \square

Remark 4.14. (1) Clearly, all objects Dx , $x \in P$ in the κ -good diagram from the proof above are κ -presentable.

(2) The limit step in the proof of 4.11 is much simpler for $\kappa = \aleph_0$ because Q is directed and thus we may take $Q^* = Q$.

In the rest of the section we investigate cellular maps and cofibrations which are small in some respect. There are two possible interpretations — either they are between κ -presentable objects or the involved transfinite composite has length $< \kappa$. We describe the relationship between these two notions of smallness.

Lemma 4.15. *Let \mathcal{K} be a cocomplete category and \mathcal{X} a class of morphisms in \mathcal{K}_κ with κ uncountable. Then $\text{cell}_\kappa(\mathcal{X}) = \kappa\text{-Tc Po}_\kappa(\mathcal{X})$.*

Proof. It is enough to show the inclusion $\text{cell}_\kappa(\mathcal{X}) \subseteq \kappa\text{-Tc Po}_\kappa(\mathcal{X})$, the opposite one is easy. Thus, let $f \in \text{cell}_\kappa(\mathcal{X})$. Following 4.13 and 4.14(1), f is the composite of a κ -good κ -directed diagram $D: P \rightarrow \mathcal{K}$ with all objects Dx κ -presentable. Then the identity on $\text{colim } D$ factors as $\text{colim } D \rightarrow Dx_1 \rightarrow \text{colim } D$ where the second morphism $\delta_{x_1}: Dx_1 \rightarrow \text{colim } D$ is the colimit cocone component for some $x_1 \in P$ and the composition $D\perp \rightarrow \text{colim } D \rightarrow Dx_1$ of f with the first morphism equals $D(\perp x_1)$. The other composition $Dx_1 \rightarrow \text{colim } D \rightarrow Dx_1$ is idempotent and gets coequalized with the identity by δ_{x_1} . Thus, there exists $x_2 \geq x_1$ such that this pair gets coequalized already by $D(x_1 x_2): Dx_1 \rightarrow Dx_2$. Proceeding inductively, we get a sequence $x_1 \leq x_2 \leq \dots$ of objects of P such that each Dx_n is equipped with an idempotent that gets coequalized with the identity by $D(x_n x_{n+1}): Dx_n \rightarrow Dx_{n+1}$. Then, it is not hard to see that $\text{colim } Dx_n \cong \text{colim } D$ and thus the composite $D\perp \rightarrow \text{colim } D$ is isomorphic to the transfinite composite of

$$D\perp \rightarrow Dx_1 \rightarrow Dx_2 \rightarrow \dots$$

Since each morphism $Dx_n \rightarrow Dx_{n+1}$ lies in $\kappa\text{-Tc Po}_\kappa(\mathcal{X})$ by Remark 4.6 (and κ -presentability of Dx_n) and κ is uncountable, the same applies to the composite. \square

Remark 4.16. When all \mathcal{X} -cofibrations are monomorphisms, the statement is true even for $\kappa = \omega$. This is because $Dx_1 \rightarrow \text{colim}_P D$ is then a monomorphism and consequently the idempotent on Dx_1 must be the identity, showing that the composite of the diagram is isomorphic already to $D\perp \rightarrow Dx_1$.

In general, the statement is not true for $\kappa = \omega$, as the following example shows.

Example 4.17. Let $\kappa = \omega$ and consider the category of modules over the ring $R = \mathbb{Z} \oplus e\mathbb{Z}$ with $e^2 = e$. Let $\mathcal{X} = \{R \xrightarrow{e} R\}$. The transfinite composite of

$$R \xrightarrow{e} R \xrightarrow{e} R \xrightarrow{e} \dots$$

is the map $R \rightarrow eR$ whose codomain is finitely presentable and annihilated by $(1 - e)$. This cannot happen in $\omega\text{-Tc Po}(\mathcal{X})$, since in any newly attached cell, there exists a non-zero element fixed by $(1 - e)$.

Lemma 4.18. *Let \mathcal{K} be a cocomplete category and \mathcal{X} a class of morphisms in \mathcal{K}_κ . Then $\text{cof}_\kappa(\mathcal{X}) = \text{Rt } \kappa\text{-Tc Po}_\kappa(\mathcal{X})$.*

Proof. The right hand side is obviously contained in the left. For the converse, let $f \in \text{cof}_\kappa(\mathcal{X})$ and express f as a retract of some $g \in \text{cell}(\mathcal{X})$,

$$\begin{array}{ccccc} B & \longrightarrow & Y & \longrightarrow & B \\ \uparrow f & & \uparrow g & & \uparrow f \\ A & \xrightarrow{\text{id}} & A & \xrightarrow{\text{id}} & A \end{array}$$

(according to 2.1(5) this retract can be taken in A/\mathcal{K}). Express g as a composite of a κ -good κ -directed diagram $D: P \rightarrow \mathcal{K}$. Since f is κ -presentable in A/\mathcal{K} , it is in fact a retract of some $A \rightarrow Dx$. Following 4.14(1), all objects Dx are κ -presentable, which finishes the proof. \square

The following lemma is essentially A.1.5.11 of [11]. Its proof only works for locally κ -presentable categories, in contrast to our previous results. We say that a diagram $D: P \rightarrow \mathcal{K}$ is κ -small, if P has $< \kappa$ objects; its composite is then said to be a κ -small composite.

Lemma 4.19. *Let \mathcal{K} be a locally κ -presentable category and \mathcal{X} a class of morphisms in \mathcal{K}_κ . Then every κ -good κ -small composite with links in $\text{Po}(\mathcal{X})$ lies in $\text{Po cell}_\kappa(\mathcal{X})$.*

Later, we will also need an obvious relative version: for an initial segment $Q \subseteq P$ such that $P \setminus Q$ has $< \kappa$ objects, the canonical map $\text{colim}_Q D \rightarrow \text{colim } D$ lies in $\text{Po cell}_\kappa(\mathcal{X})$.

Intuitively, the lemma says that the effect of attaching $< \kappa$ cells to an object takes place in some κ -presentable part. Attaching the cells solely to this small part results in a cellular map between κ -presentable objects with the original map being its pushout.

Proof. Let $D: P \rightarrow \mathcal{K}$ be a κ -good κ -small diagram with links in $\text{Po}(\mathcal{X})$. Express the bottom object of the composite $D \perp \rightarrow \text{colim } D$ as a κ -filtered colimit $D \perp = \text{colim}_{i \in \mathcal{I}} A_i$ of a diagram $A: \mathcal{I} \rightarrow \mathcal{K}$ of κ -presentable objects such that \mathcal{I} has κ -small colimits and A preserves them. For instance, we can take the canonical diagram $\mathcal{I} = \mathcal{K}_\kappa / D \perp$ and its projection A sending $X \rightarrow D \perp$ to X .

We will construct inductively a smooth chain $i_Q \in \mathcal{I}$, indexed by initial segments Q of (P, \preceq) as in Proposition 4.5, whose images under A are denoted $A_Q = A i_Q$, and morphisms $f_Q: A_Q \rightarrow B_Q$ in $\text{cell}_\kappa(\mathcal{X})$ such that $D \perp \rightarrow \text{colim}_Q D$ is a pushout of f_Q along the component

$A_Q \rightarrow D \perp$ of the colimit cocone,

$$\begin{array}{ccc} D \perp & \longrightarrow & \operatorname{colim}_Q D \\ \uparrow & & \downarrow \lrcorner \\ A_Q & \xrightarrow{f_Q} & B_Q \end{array}$$

These data are subject to the following two conditions:

- (1) for a successor $Q' \subseteq Q$, the morphism f_Q is a composition of the pushout of $f_{Q'}$ along the obvious morphism $A_{Q'} \rightarrow A_Q$ with some element of $\operatorname{Po}(\mathcal{X})$;

$$(\star) \quad \begin{array}{ccccccc} & & & f_Q & & & \\ & & & \curvearrowright & & & \\ & A_Q & \longrightarrow & A_Q \sqcup_{A_{Q'}} B_{Q'} & \longrightarrow & B_Q & \\ & \uparrow & & \downarrow \lrcorner & & \downarrow \lrcorner & \\ A_{Q'} & \xrightarrow{f_{Q'}} & B_{Q'} & & X & \xrightarrow{g_Q} & Y \end{array}$$

- (2) for a limit Q , the morphism $A_Q \rightarrow B_Q$ is a colimit of the pushouts of $A_{Q'} \rightarrow B_{Q'}$ over $Q' \subseteq Q$. (In particular, it is a transfinite composite of pushouts of the $g_{Q'}$ with $Q' \subsetneq Q$.)

For $Q = P$ we obtain $D \perp \rightarrow \operatorname{colim} D$ as a pushout of $A_P \rightarrow B_P$ that lies in $\operatorname{cell}_\kappa(\mathcal{X})$ as desired.

Since the limit steps are determined by condition (2), it remains to describe the successor case. Let $Q' \subseteq Q$ be successive initial segments and assume that the only element x of $Q' \setminus Q$ is isolated — otherwise, we may take $f_Q = f_{Q'}$. By induction, the partial composite $D \perp \rightarrow \operatorname{colim}_{Q'} D$ is a pushout of $f_{Q'}: A_{Q'} \rightarrow B_{Q'}$ lying in $\operatorname{cell}_\kappa(\mathcal{X})$. Then $\operatorname{colim}_{Q'} D$ is a colimit of the κ -filtered diagram of pushouts $A_j \sqcup_{A_{Q'}} B_{Q'}$, indexed by arrows $i_{Q'} \rightarrow j$. The morphism $\operatorname{colim}_{Q'} D \rightarrow \operatorname{colim}_Q D$ is a pushout of $Dx^- \rightarrow Dx$ and thus a pushout of some $X \rightarrow Y$ in \mathcal{X} . The attaching map $X \rightarrow Dx^- \rightarrow \operatorname{colim}_{Q'} D$ factors through some $A_j \sqcup_{A_{Q'}} B_{Q'}$. We set $i_Q = j$ and obtain f_Q as in (\star) . \square

Corollary 4.20. *Let \mathcal{K} be a locally κ -presentable category and \mathcal{X} a class of morphisms in \mathcal{K}_κ . Then*

$$\kappa\text{-TcPo}(\mathcal{X}) = \operatorname{Po} \kappa\text{-TcPo}_\kappa(\mathcal{X}).$$

Proof. The right hand side is clearly contained in the left. The opposite inclusion is easily implied by the previous lemma. \square

5. APPLICATIONS

An object K of a category \mathcal{K} is called \mathcal{X} -*cofibrant* if the unique morphism $0 \rightarrow K$ from an initial object belongs to $\text{cof}(\mathcal{X})$; \mathcal{X} -*cellular* objects are defined analogously.

Corollary 5.1. *Let \mathcal{K} be a cocomplete category and \mathcal{X} a class of morphisms in \mathcal{K}_κ . Then any \mathcal{X} -cofibrant object of \mathcal{K} is a κ -filtered colimit of κ -presentable \mathcal{X} -cofibrant objects.*

Proof. For cellular objects the claim follows from 4.13 and 4.14(1). Since any cofibrant object is a retract of a cellular one, the result follows from [13] 2.3.11. (the proof applies in the case when \mathcal{K} is not locally κ -presentable but merely cocomplete). \square

Corollary 5.2. *Let κ be an uncountable regular cardinal, \mathcal{K} a locally κ -presentable category and \mathcal{X} a class of morphisms in \mathcal{K}_κ . Then any \mathcal{X} -cofibrant object of \mathcal{K} is a κ -good κ -directed colimit of κ -presentable \mathcal{X} -cofibrant objects where all links are \mathcal{X} -cofibrations.*

Proof. The proof is the same as in Corollary 5.1 but we use Theorem B.1 instead of [13] 2.3.11. \square

Remark 5.3. (1) According to the proof of Corollary 5.2, the links even lie in $\text{Po cof}_\kappa(\mathcal{X})$.

(2) Let \mathcal{K} be the category of modules over a ring R and let \mathcal{P} be the class of projective R -modules. A monomorphism $f: A \rightarrow B$ is called a \mathcal{P} -monomorphism if its cokernel is a projective module P . Then f is the coproduct injection $A \rightarrow A \oplus P$. We get a weak factorization system $(\mathcal{P}\text{-Mono}, \text{Epi})$ whose left class consists of all \mathcal{P} -monomorphisms and the right class of all epimorphisms. The left class is cofibrantly generated by a morphism $i: 0 \rightarrow R$. Cofibrant objects are precisely projective modules. Following 5.2, every projective module is a ω_1 -good ω_1 -directed colimit of ω_1 -presentable projective modules where all links are \mathcal{P} -monomorphisms. Hence all morphisms of the corresponding diagram are coproduct injections. Thus every projective module is a coproduct of countably generated projective modules, which is a classic theorem due to Kaplansky.

This also shows that Corollary 5.2 cannot be extended to ω because there exist rings which admit projective modules which are not coproducts of finitely generated projective modules (see [8] 7.15).

Let \mathcal{K} be a Grothendieck category. Given a class \mathcal{S} of objects of \mathcal{K} , a monomorphism f is called an \mathcal{S} -monomorphism if its cokernel belongs to \mathcal{S} . An object K is \mathcal{S} -filtered if the unique morphism $0 \rightarrow K$ is a

transfinite composite of \mathcal{S} -monomorphisms. A class \mathcal{C} is *deconstructible* if it is the class of \mathcal{S} -filtered objects for a set \mathcal{S} (see [16]).

Remark 5.4. A class \mathcal{C} is deconstructible if and only if \mathcal{C} -monomorphisms are the cellular closure of a set of morphisms.

Sufficiency is easy because if \mathcal{C} -monomorphisms are $\text{cell}(\mathcal{X})$ for a set \mathcal{X} then the set \mathcal{S} of cokernels of morphisms from \mathcal{X} makes \mathcal{C} deconstructible. Necessity is [17], Proposition 2.7.

Remark 5.5. Let R be a ring and $(\mathcal{A}, \mathcal{B})$ a cotorsion pair of finite type, i.e., generated by a set \mathcal{S} of finitely presentable R -modules. Any $A \in \mathcal{S}$ is a quotient $p_A: A^* \rightarrow A$ of a free module and $\ker(p_A)$ is a morphism between finitely presentable modules. Then \mathcal{A} -monomorphisms are cellularly generated by $\ker(p_A)$, $A \in \mathcal{S}$ (see [17] as above). Following Corollary 5.1, any module from \mathcal{A} is a directed colimit of finitely presentable modules from \mathcal{A} . This fact was proved in [7] 2.3.

An object K of a model category \mathcal{K} is called *acyclic* if $K \rightarrow 1$ is a weak equivalence.

Corollary 5.6. *Let \mathcal{K} be a κ -combinatorial model category where 1 is κ -presentable and any morphism $K \rightarrow 1$ splits by a cofibration. Then any acyclic object of \mathcal{K} is a κ -directed colimit of κ -presentable acyclic objects.*

Proof. Let $\mathcal{K}_* = 1 \downarrow \mathcal{K}$ be the associated pointed model category (see [6] 1.1.8). Let K be an acyclic object of \mathcal{K} . Following our assumption, there is a cofibration $f: 1 \rightarrow K$ and, since K is acyclic, f is a trivial cofibration. Thus (K, f) is trivially cofibrant in \mathcal{K}_* (see [6] 1.1.8). Since \mathcal{K}_* is κ -combinatorial as well, 5.1 applied to trivial cofibrations in \mathcal{K}_* yields that (K, f) is a κ -directed colimit of trivially cofibrant κ -presentable objects (K_i, f_i) . Since 1 is κ -presentable, any K_i is κ -presentable in \mathcal{K} . Since each f_i is a trivial cofibration in \mathcal{K} , each K_i is acyclic in \mathcal{K} . \square

Remark 5.7. In particular, any acyclic simplicial set is a directed colimit of finitely presentable acyclic simplicial sets (see [9] 6.3).

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APPENDIX A. GENERAL FAT SMALL OBJECT ARGUMENT

In this section, let κ be an arbitrary regular cardinal and let \mathcal{X} be a set of morphisms in \mathcal{K}_κ . We will describe in this appendix an alternative to the usual small object argument.

Theorem A.1. *Let $f: A \rightarrow B$ be a morphism. Then, there exists a factorization of f , whose left part lies in $\kappa\text{-GdDirPo}(\mathcal{X})$ and whose right part lies in \mathcal{X}^\square .*

Proof. The factorization is obtained as a transfinite iteration of length κ of the construction, that (similarly to the usual small object argument) adds cells that solve all the possible lifting problems. Thus, we consider by induction, for $\alpha < \kappa$, a factorization

$$A \rightarrow \operatorname{colim} D_\alpha \rightarrow B$$

where $D_\alpha: P_\alpha \rightarrow K$ is a κ -good κ -directed diagram with links in $\operatorname{Po}(\mathcal{X})$. We assume, that for each $\beta < \alpha$, P_β is an initial segment in P_α and that D_β is the restriction of D_α . Consider the set of all squares

$$\begin{array}{ccc} \operatorname{colim} D_\alpha & \xrightarrow{f} & B \\ x \uparrow & & \uparrow y \\ X & \xrightarrow{g} & Y \end{array}$$

parametrized by x, y , and for each such square, choose a factorization of x through some $X \rightarrow D_\beta$. Then form the pushout square

$$\begin{array}{ccc} D_\beta & \longrightarrow & D_{x,y} \\ \uparrow & \lrcorner & \uparrow \\ X & \longrightarrow & Y \end{array}$$

Next, add to P_α , for each x, y , objects $p_{x,y}$ with $p_{x,y} > \beta$ and extend the diagram D_α to $p_{x,y}$ by $D_{x,y}$. Finally, to obtain $D_{\alpha+1}: P_{\alpha+1} \rightarrow K$, perform the $*$ -construction of Lemma 4.10. In the limit steps, take P_α to be the $*$ -construction of the union $\bigcup_{\beta < \alpha} P_\beta$.

Thus, in each step $\alpha \leq \kappa$, we obtain a κ -good κ -directed diagram $D_\alpha: P_\alpha \rightarrow K$ with links in $\operatorname{Po}(\mathcal{X})$. The factorization is

$$A \rightarrow \operatorname{colim} D_\kappa \rightarrow B.$$

It is easy to see that $\operatorname{colim} D_\kappa \rightarrow B$ lies in \mathcal{X}^\square — since any $X \rightarrow \operatorname{colim} D_\kappa$ factors through some $\operatorname{colim} D_\alpha$ with $\alpha < \kappa$, the lifting problem is solved in $\operatorname{colim} D_{\alpha+1}$. \square

Remark A.2. There is a slight difference between the proof of this theorem and a direct application of Theorem 4.11 to the usual small object argument: here we attach a number of cells at once and only then apply the $*$ -construction. We could have developed the rearrangement in Theorem 4.11 for transfinite composites of pushouts of *coproducts* of

morphisms in \mathcal{X} which would have given this exact version of the small object argument.

APPENDIX B. ELIMINATION OF RETRACTS

Retracts of $A \rightarrow B$ in this section are to be understood as retracts in the category A/\mathcal{K} . These are enough to produce all cofibrations as retracts of cellular morphisms, as explained in 2.1(5). Moreover we assume that \mathcal{K} is locally κ -presentable and that \mathcal{X} is a set of morphisms in \mathcal{K}_κ for some *uncountable* cardinal κ .

The following theorem shows, that the use of retracts is unnecessary, at least if one is willing to enlarge the generating set \mathcal{X} . This result was proved by Lurie in [11], A.1.5.12. Here we present an alternative proof, that relies on κ -good κ -directed diagrams.

Theorem B.1. $\text{cof}(\mathcal{X}) = \text{cell}(\text{cof}_\kappa(\mathcal{X}))$.

We start with a couple of generalities.

Let P be a κ -good κ -directed poset. We say, that an upper bound x of an initial segment $Q \subseteq P$ is a *strong upper bound*, if all the elements in $(\downarrow x) \setminus Q$ are limit. Equivalently¹, the canonical map $\text{colim}_Q D \rightarrow Dx$ is an isomorphism for all smooth diagrams $D: P \rightarrow \mathcal{K}$. We define

$$\overline{Q} = \{x \in P \mid x \text{ is a strong upper bound of some subset } R \subseteq Q\}.$$

By its definition, the closure \overline{Q} is an initial segment and, according to Lemma 4.9, the canonical map $\text{colim}_Q D \rightarrow \text{colim}_{\overline{Q}} D$ is an isomorphism. We say that an initial segment Q is *closed*, if $\overline{Q} = Q$.

It is obvious from its construction in Lemma 4.10 that P^* has strong upper bounds of all κ -small initial segments.

Lemma B.2. *Let P be a κ -good poset with strong upper bounds of all κ -small initial segments. Let $D: P \rightarrow \mathcal{K}$ be a smooth diagram whose all objects are κ -presentable and let there be given an idempotent $f: \text{colim } D \rightarrow \text{colim } D$ in $(D\perp)/\mathcal{K}$.*

Then there exists an endofunctor $S: P \rightarrow P$ with $S\perp = \perp$ and $x \leq Sx$, and an idempotent natural transformation $\varphi: DS \rightarrow DS$ with

¹If x is a strong upper bound of Q then the inclusion $Q \subseteq (\downarrow x)$ induces an isomorphism $\text{colim}_Q D \rightarrow \text{colim}_{\downarrow x} D \cong Dx$ by Lemma 4.9. In the opposite direction, we observe first that a representable functor $D = P(y, -): P \rightarrow \mathbf{Set}$ is smooth if and only if y is isolated. Thus, if an isolated $y \in (\downarrow x) \setminus Q$ existed, we would then get $D|_Q = \emptyset$ and $Dx = 1$, a contradiction.

$\varphi_\perp = \text{id}$, such that the following diagram commutes.

$$\begin{array}{ccc} \text{colim } D & \xrightarrow{f} & \text{colim } D \\ \cong \downarrow & & \downarrow \cong \\ \text{colim } DS & \xrightarrow{\varphi_*} & \text{colim } DS \end{array}$$

(The vertical maps are induced by $D(x, Sx): Dx \rightarrow DSx$.)

Moreover, if $Q \subseteq P$ is a closed initial segment and the idempotent f extends an idempotent $f': \text{colim}_Q D \rightarrow \text{colim}_Q D$, and if there are given the $S': Q \rightarrow Q$ and $\varphi': DS' \rightarrow DS'$ as above for f' , then the S and φ may be constructed as extensions of S' and φ' .

This lemma roughly says that there are many objects in the diagram with idempotents on them (they are cofinal in P). If these constituted a good diagram, we could have used them to express the image of the idempotent f as a good colimit of retracts. This is however not true in general and that is why we need the relative version.

Proof. The basic idea is rather simple. We construct Sx as a strong upper bound of a chain $S_1x \leq S_2x \leq \dots$ and φ_x as a colimit of morphisms $(\varphi_n)_x$ in the diagram

$$\begin{array}{ccccccc} DS_1x & \longrightarrow & DS_2x & \longrightarrow & \dots & \longrightarrow & DSx & \longrightarrow & \text{colim}_P D \\ & \searrow (\varphi_1)_x & & \searrow (\varphi_2)_x & & & \downarrow \varphi_x & & \downarrow f \\ DS_1x & \longrightarrow & DS_2x & \longrightarrow & \dots & \longrightarrow & DSx & \longrightarrow & \text{colim}_P D \end{array}$$

where the unnamed arrows are induced by D from the unique arrows in P . We will stick to this convention in the rest of the proof.

The S_nx and $(\varphi_n)_x$ are constructed inductively. Without the requirement of naturality, they are obtained by factoring

$$DS_nx \rightarrow \text{colim } D \xrightarrow{f} \text{colim } D$$

as $DS_nx \xrightarrow{(\varphi_n)_x} DS_{n+1}x \rightarrow \text{colim } D$. By choosing $S_{n+1}x$ big enough, we may assume that $S_{n+1}x \geq S_nx$ and that the following compositions are equal

$$(\diamond) \quad \begin{array}{ccccc} & & DS_nx & & \\ & \nearrow (\varphi_{n-1})_x & & \searrow & \\ DS_{n-1}x & \xrightarrow{(\varphi_{n-1})_x} & DS_nx & \xrightarrow{(\varphi_n)_x} & DS_{n+1}x \\ & \searrow & & \nearrow (\varphi_n)_x & \\ & & DS_nx & & \end{array}$$

ensuring that φ_x will be idempotent.

To ensure naturality, we have to construct $(\varphi_n)_x$ inductively with respect to x . We set $S_{n+1}\perp = \perp$ and $(\varphi_n)_\perp = \text{id}$. Assume, that $(\varphi_n)_y$ has been defined for all $y < x$. We thus have a diagram

$$\begin{array}{ccc}
 \text{colim } D & \xrightarrow{f} & \text{colim } D \\
 \uparrow & & \uparrow \\
 DS_n x & \xrightarrow{(\varphi_n)_x} & DS_{n+1} x \\
 \uparrow & & \uparrow \\
 \text{colim}_{y < x} DS_n y & \xrightarrow{g} & Dz
 \end{array}$$

(Note: The diagram shows a commutative square with a dashed diagonal arrow from $DS_n x$ to Dz and a dashed diagonal arrow from $DS_{n+1} x$ to $\text{colim } D$.)

where z is an arbitrary upper bound of the set $\{S_{n+1}y \mid y < x\}$ and the bottom map g is given on $DS_n y$ as the composition

$$DS_n y \xrightarrow{(\varphi_n)_y} DS_{n+1} y \rightarrow Dz.$$

The solid square commutes and it is easy to find some $S_{n+1}x$ and a factorization $(\varphi_n)_x$ using the κ -presentability of $DS_n x$ and $\text{colim}_{y < x} DS_n y$. In this way both S_{n+1} will be a functor and φ_n a natural transformation. We may still achieve the commutativity of (\diamond) by passing to a bigger $S_{n+1}x$.

Similarly, the strong upper bound Sx of the chain $S_1x \leq S_2x \leq \dots$ is chosen inductively, starting with $S\perp = \perp$. When all the Sy have been chosen for $y < x$, Sx is chosen as a strong upper bound for the initial segment spanned by S_1x, S_2x, \dots and all the Sy with $y < x$. At the same time Sx is a strong upper bound of the initial segment spanned by the S_1x, S_2x, \dots since all the Sy , $y < x$, lie in the closure of this initial segment, so that Lemma 4.9 applies.

When S' and φ' are given, we may choose $S_{n+1}x = S'x$, $(\varphi_n)_x = \varphi'_x$, and $Sx = S'x$ in the above, whenever $x \in Q$. \square

Proof of Theorem B.1. Suppose that $A \rightarrow B$ is a cellular map and that a retract of it is given by an idempotent $f: B \rightarrow B$ in A/\mathcal{K} . We write $A \rightarrow B$ as a colimit of a κ -good κ -directed diagram $D: P \rightarrow A/\mathcal{K}$ with links in $\text{Po}(\mathcal{X})$. By our assumptions, it consists of κ -presentable objects of A/\mathcal{K} and, applying the $*$ -construction of Lemma 4.10 if necessary, we may construct D in such a way that strong upper bounds of all κ -small initial segments exist. Thus, Lemma B.2 is applicable to D .

We may construct the colimit $\text{colim } D$ inductively similarly to the proof of Proposition 4.5. It will be a transfinite composite of partial

colimits $\text{colim}_{P_i} D$ equipped with compatible idempotents

$$f_i: \text{colim}_{P_i} D \rightarrow \text{colim}_{P_i} D,$$

where the P_i form a transfinite sequence of closed initial segments with respect to the inclusion.

We start with $P_0 = \{\perp\}$ and $f_0 = \text{id}_{D\perp}$.

Suppose, that we have constructed P_i and f_i . Then, we construct a new endofunctor S and a natural transformation φ on P by Lemma B.2 by first constructing them on P_i and then extending to P . Next, take any minimal element x not in P_i and denote by Q the initial segment generated by the sequence Sx, S^2x, \dots ; it is κ -small. Then the colimit of D over $P_i \cup Q$ can be written as the pushout

$$\begin{array}{ccccc} \text{colim}_{P_i} D & \longrightarrow & \text{colim}_{P_i \cup Q} D & \xrightarrow{\cong} & \text{colim}_{P_{i+1}} D \\ \uparrow & & \downarrow \lrcorner & \uparrow & \\ \text{colim}_{P_i \cap Q} D & \longrightarrow & \text{colim}_Q D & & \end{array}$$

Finally we take $P_{i+1} = \overline{P_i \cup Q}$. This does not change the colimit. The functor S preserves both P_i and Q by construction and thus also $P_i \cap Q$ and $P_i \cup Q$. Therefore, we have idempotents on all colimits in the above square and they are compatible; we denote that on $\text{colim}_{P_{i+1}} D$ by f_{i+1} .

Finally, we have to explain how to compute the retract of the composite $D\perp \rightarrow \text{colim}_P D$. We have split this composite into a transfinite composite in the category of objects with idempotents. For each i , let E_i denote the image of the idempotent on $\text{colim}_{P_i} D$ with the retraction r_i . Then consider the following pushout (which simply defines F_i)

$$\begin{array}{ccc} E_i & \longrightarrow & F_i \\ r_i \uparrow & & \downarrow \lrcorner \uparrow \\ \text{colim}_{P_i} D & \longrightarrow & \text{colim}_{P_{i+1}} D \end{array}$$

There is an induced idempotent on F_i , which restricts to id on E_i and whose image is exactly E_{i+1} . At the same time, $E_i \rightarrow F_i$ is a pushout of the map $\text{colim}_{P_i \cap Q} D \rightarrow \text{colim}_Q D$ which, by κ -smallness of Q and a relative version of Lemma 4.19, is a pushout of an element of $\text{cell}_\kappa(\mathcal{X})$. By Lemma B.3 below, $E_i \rightarrow E_{i+1}$ lies in $\text{Po cof}_\kappa(\mathcal{X})$. Thus, the composite $A \rightarrow \text{colim}_i E_i$, i.e. the image of the idempotent f that we started with, is $\text{cof}_\kappa(\mathcal{X})$ -cellular. \square

The following lemma is A.1.5.10 of [11].

Lemma B.3. $\text{Rt Po cell}_\kappa(\mathcal{X}) \subseteq \text{Po cof}_\kappa(\mathcal{X})$.

Proof. Let $X \rightarrow Y$ be a pushout of a morphism $A \rightarrow B$ from \mathcal{K}_κ and let $f: Y \rightarrow Y$ be an idempotent in X/\mathcal{K} . We want to express its image as an element of $\text{Pocof}_\kappa(\mathcal{X})$. Write X as the canonical κ -filtered colimit $X = \text{colim } X_\alpha$ of κ -presentable objects of A/\mathcal{K} . Corresponding to this, Y is a colimit $Y = \text{colim } Y_\alpha$, where $Y_\alpha = X_\alpha \sqcup_A B$. As this diagram has all κ -small colimits, we may use the proof of Lemma B.2 to find a chain $\alpha_1 \rightarrow \alpha_2 \rightarrow \cdots$ together with morphisms $\varphi_n: Y_{\alpha_n} \rightarrow Y_{\alpha_{n+1}}$ that induce an idempotent f_α on $Y_\alpha = \text{colim}_n Y_{\alpha_n}$. Since f restricts to id on X , we may assume at each point, that φ_n restricts to the map $X_{\alpha_n} \rightarrow X_{\alpha_{n+1}}$ in the canonical diagram (by passing to “bigger” α_{n+1} if necessary). In this way, f_α will be an idempotent in X_α/\mathcal{K} . Then the image of f is a pushout of the image of f_α , as required. \square

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